Recall:

Theorem If $H \trianglelefteq G$ (normal subgroup)

can construct factor group G/H as follows

$G/H = \{aH, \ a \in G\}$ left cosets of H

operation: $(aH)(bH) = abH$

Proof already showed

operation is well-defined

(i.e. if $aH = a'H$, $bH = b'H$

$$\Rightarrow (a'H)(b'H) = (aH)(bH)$$

$$a'b'H = abH$$

(Remark: not true for non-normal subgroups!)

associativity: $(aH\, bH)\, cH = abH\, cH = (ab)cH$

$$aH\,(bH\,cH) = aH\, bcH = a(bc)H$$ ✓

identity element: $eH = H$

inverse: $(aH)^{-1} = a^{-1}H$ (check for yourself!)

Examples:

(add. notation!)

① $G = (\mathbb{Z}, +)$ $H = (4\mathbb{Z}, +)$

$$G/H = \{H,\ 1+H,\ 2+H,\ 3+H\}$$

$H = 4\mathbb{Z}$

$$(2+H) + (3+H) = 5 + H = 1 + H$$

$H = 4\mathbb{Z}$, $H = 4\mathbb{Z}$

Remark: Can show:

$\Phi: \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}_4$, $j + 4\mathbb{Z} \to j$ defines an isomorphism!
$j \in \{0,1,2,3\}$

Remark: A_4 is not abelian,
but its factor group $A_4/H \cong \mathbb{Z}_3$
IS abelian.

more examples later.

Ch 8 External Direct Products

Idea: Given 2 or more groups
$\Rightarrow$ build bigger group.

Def. Let $G_1, G_2, \ldots G_n$ be groups
Then the external direct product $G_1 \oplus G_2 \oplus \ldots \oplus G_n$
consists of all elements of the form $(g_1, g_2, g_3, \ldots, g_n)$
where $g_i \in G_i$, $i = 1, 2, \ldots, n$.

Theorem The external direct product $G_1 \oplus \dots \oplus G_n$ forms a group under coordinate wise group operation

i.e. $(g_1, \dots g_n) \cdot (h_1, \dots h_n) = (g_1 h_1, g_2 h_2, \dots g_n h_n)$

proof exercise

e.g. if $e_i \in G_i$ is the identity element of G_i

$(e_1, e_2 \dots, e_n)$ is " " " " $G_1 \oplus \dots \oplus G_n$

Examples: ① $U(6) \oplus U(8)$

$U(6) = \{1, 5\}$

$U(8) = \{1, 3, 5, 7\}$

$\Rightarrow U(6) \oplus U(8) = \{ (1,1), (1,3), (1,5), (1,7),$
$(5,1), (5,3), (5,5), (5,7) \}$

Ex: $(5,5) \cdot (5,7) = (\underset{\text{mod } 6}{25}, \underset{\text{mod } 8}{35}) = (1, 3)$

(2) $\mathbb{Z}_2 \oplus \mathbb{Z}_3 = \{ (0,0), (0,1), (0,2), (1,0), (1,1), (1,2) \}$

additive notation

<u>Question</u>: Is this a new group of order 6 up to isomorphism.

so far we know $\underbrace{2}_{\text{non-isomorphic}}$ groups of order 6, $\mathbb{Z}_6$ and S_3

Is $\mathbb{Z}_2 \oplus \mathbb{Z}_3$ isom. to $\mathbb{Z}_6$?

$\Longleftrightarrow$ Can we find an element of order 6 in $\mathbb{Z}_2 \oplus \mathbb{Z}_3$?

check order for each element.

try. $(1,1)$.

add. notations

$$\left.\begin{aligned} 2(1,1) &= (2,2) = (0,2) \\ 3(1,1) &= (3,3) = (1,0) \end{aligned}\right\} \Longrightarrow \operatorname{ord}(1,1) = 6$$

mod 2, mod 3

reason: $\operatorname{ord}(1,1) \mid 6$

have excluded $1, 2, 3$

$\Rightarrow \operatorname{ord}(1,1) = 6.$

Conclusion: $\mathbb{Z}_2 \oplus \mathbb{Z}_3 \cong \mathbb{Z}_6$

isom given by $j \in \mathbb{Z}_6 \longrightarrow j\,(1,1) = (j,j) \in \mathbb{Z}_2 \oplus \mathbb{Z}_3$

(again: $\operatorname{ord}(1,1) = 6 \Rightarrow \langle (1,1) \rangle = \mathbb{Z}_2 \oplus \mathbb{Z}_3$ as $|\mathbb{Z}_2 \oplus \mathbb{Z}_3| = 6$.)

③ $\mathbb{Z}_2 \oplus \mathbb{Z}_2 = \{(0,0), \underbrace{(1,0), (0,1), (1,1)}_{\text{all have order 2}}\}$

check:

$\Rightarrow$ $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ is a group of order 4
but it does NOT contain an element of order 4

$\Rightarrow$ $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ is NOT isomorphic to $\mathbb{Z}_4$

Remark: One can show that $\mathbb{Z}_4$ and $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ are the only groups of order 4 up to isom.

i.e. any group G with $|G| = 4$ must be isom. to $\mathbb{Z}_4$ or to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Useful Theorem:

Theorem: Let G be a group with $a^2 = e$ for all $a \in G$

$\Rightarrow$ G is abelian, i.e. $ab = ba$ for all $a, b \in G$

proof. $a, b \in G \Rightarrow ab \in G \Rightarrow \boxed{(ab)^2 = e}$

$$\boxed{a^2 b^2 = e \cdot e = e}$$

$\Rightarrow$ $a^2 b^2 = e = (ab)^2 = abab$ left cancel.

$\Rightarrow$ $ab^2 = bab$ right cancellation

$ab = ba$ ✓

2. $G = \mathbb{Z}_{12}$ $H = \langle 3 \rangle \subset \mathbb{Z}_{12}$

$H = \{0, 3, 6, 9\}$

we have 3 cosets

$H = \{0, 3, 6, 9\}$

$1 + H = \{1, 4, 7, 10\}$

$2 + H = \{2, 5, 8, 11\}$

Can show: $G/H \cong \mathbb{Z}_3$

3. let $H = \{id, (12)(34), (13)(24), (14)(23)\} \subset A_4$

have shown: $H \trianglelefteq A_4$

$\Rightarrow$ can form factor group A_4/H

$|A_4/H| = 3 \Rightarrow A_4/H \cong \mathbb{Z}_3$

$A_4/H = \{H, (123)H, (132)H\}$

check: $(132) \notin (123)H$

$(132) = (123)^2$